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On the complete solution of a perturbed topological conformal field theory at $c = 3$

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Abstract. The problem of determination of the *flat coordinates* for a model of a topological conformal field theory corresponding to the 'twisted' version of an $N = 2$ superconformal Landau–Ginzburg field theory with central charge $c = 3$ is analysed here. The model is characterized by a Landau–Ginzburg superpotential of the form: $W = \frac{1}{4}x^4 + \frac{1}{4}y^4$. All possible relevant and marginal perturbations with their corresponding couplings (expressed as functions of the dimensionless flat-coordinate), are added to the above superpotential to give us the perturbed topological field theory model. It is seen that the couplings can be completely determined (and hence also the dependence of the perturbations on the flat coordinate) by imposing the conditions *of flatness* on the space of couplings of the perturbed theory.

1. Introduction and formulation of the problem

Topological Landau–Ginzburg theories [1,2], i.e. the 'twisted' [3] version of the chiral primary subsector of the class of $N = 2$ superconformal field theories described by a Landau–Ginzburg superpotential, have been of much recent interest—not only in their own right [4], but also because they completely determine the modular dependence of the Yukawa couplings in string theories [5]. Their study has considerably widened our knowledge of the interrelations of certain aspects of singularity theory [6] and Picard–Fuchs theory of differential equations for the periods of holomorphic forms [7,8]. The correlation functions of such topological models are completely determined by a prepotential (the free energy) $\mathcal{F}$. In particular, there exists a special set of *flat* coordinates, t_i, in terms of which the three-point correlation function C_{ijk} can be expressed [1] in the form

$$C_{ijk} = \frac{\partial^3 \mathcal{F}(\underline{t})}{\partial t_i \partial t_j \partial t_k} \tag{1.1}$$

where the associativity constraint implies [1,9]

$$\sum_m C_{ij}{}^m C_{klm} = \sum_m C_{il}{}^m C_{kjm}. \tag{1.2}$$

Further, there exists a unique coordinate t_0 such that the metric η (which has been used for the raising and lowering of indices in the above equation) on the space of the topological field theory, defined by the two-point function can be expressed as [1]

$$\eta_{ij} = \frac{\partial^3 \mathcal{F}(\underline{t})}{\partial t_i \partial t_j \partial t_0}.$$

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These coordintes t_i are called *flat* when the two-point function η_{ij} (defined above), is an invertible t-independent matrix—thus providing us with a natural, flat metric on the space of chiral primary fields [1, 10].

Let us now consider the following model of a topological conformal field theory (TCFT) corresponding to a twisted version of an $N = 2$ superconformal field theory defined by the quasi-homogeneous superpotential [6, 11]

$$W_0(X) = \tfrac{1}{4}x^4 + \tfrac{1}{4}y^4$$

where $x(z, \overline{z})$ and $y(z, \overline{z})$ are Landau–Ginzburg fields with $U(1)$ charges given by

$$q_x = q_y = \tfrac{1}{4}.$$

The above superpotential corresponds to an $N = 2$ superconformal field theory (SCFT) with the central charge c given by [9]

$$c = 6\sum_i \left(\frac{1}{2} - q_i\right) = 3$$

and with an underlying (associative) chiral ring structure which is essentially isomorphic to the multiplicative polynomial ring generated by the basis [9]

$$\mathcal{R} = \{1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2\}.$$

The dimension μ of the ring is given by [9]

$$\mu = \prod_i \left(\frac{1}{q_i} - 1\right) = 9.$$

The Hessian of the above superpotential is given by

$$h_0(x, y) = \det\left|\frac{\partial^2 W_0}{\partial x_i \partial x_j}\right| = 9x^2y^2.$$

It is clearly non-degenerate at the critical points of W_0 and coincides (apart from a numerical normalization factor) with the unique chiral field with the maximal $U(1)$ charge. The $U(1)$ charges of the elements of the basis are given by $q_\alpha = \{0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, 1\}$. The unique chiral field with the maximal $U(1)$ charge is x^2y^2. All correlation functions of the unperturbed theory must obey the $U(1)$ charge conservation. Thus a correlation function of the chiral primaries $\langle X_{i_1} X_{i_2} \ldots\ldots X_{i_n}\rangle$ will vanish identically, unless

$$\sum_{i=1}^{n} q_i = \frac{c}{3} = 1.$$

Among these correlation functions, of particular importance are the two- and three-point correlation functions. In our particular case, the two-point function $\langle x^2y^2\rangle$, which is also identically equal to the expectation value of the chiral field with the highest $U(1)$ charge, defines the metric η.

Let us now consider perturbing the superpotential in the most general way [1, 12]. In order that the singularity structure of the superpotential is not altered, we must perturb the theory with fields with $U(1)$ charges $q_i \leqslant 1$ [9, 13], and these therefore correspond to the relevant and marginal perturbations only. In the context of the $N = 2$ theory this implies that the perturbing fields must be linear combinations of our basis fields in $\mathcal{R}$. Introducing couplings t_α for each of the fields X_α, the most general form of the perturbed superpotential may be written as [1]

$$W(X, t_\alpha) = W_0 + \sum_{\alpha=1}^{9} t_\alpha X_\alpha \tag{1.3}$$

where the polynomials X_α form the basis of $\mathcal{R}$. The perturbed fields

$$\phi_\alpha = \partial W / \partial t_\alpha$$

satisfy the same multiplicative ring algebra [1, 9]

$$\left(\frac{\partial W}{\partial t_j}\right)\left(\frac{\partial W}{\partial t_k}\right) = \sum_i C^i{}_{jk}\left(\frac{\partial W}{\partial t_i}\right) \qquad (\text{mod } \vec{\nabla}_x W).$$

The coefficients $C^i{}_{jk}$ form an associative ring algebra (which is, however, no longer nilpotent). One can now define the tensor

$$\eta_{ij} = \text{residue}\left[\frac{\phi_i(t)\phi_j(t)}{(\partial_x W)(\partial_y W)}\right].$$

Then it can be seen [1, 10]:

(i) that the tensor η is non-degenerate and can be considered as defining a metric; and

(ii) that the metric η has zero curvature, and hence there exists a canonical coordinate system (defined modulo linear transformations with constant coefficients) in which the metric is constant (and hence independent of t).

The perturbed superpotential is no longer quasihomogeneous and hence now the $U(1)$ charge conservation no longer holds. The $U(1)$ charges associated with the couplings $(t_1, t_2, \ldots t_9)$ are $(1, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$ respectively. The coupling constant t_9 associated with the marginal perturbation x^2y^2 is dimensionless and is of special significance as it provides us with a dimensionless 'flat' coordinate, for which we introduce the special notation $t_9 = t$. The perturbing couplings t_α can be regarded as coordinates in the coupling constant space. The family of TCFT described by W in equation (1.3) corresponds to versal deformation in singularity theory, with t_α being the parameters of deformation; the point $(t_1, t_2, \ldots t_9) = (0, 0, \ldots 0)$ corresponding to the unperturbed theory. The coordinates t_α form a distinguished basis in the space of couplings in that they correspond to the directions that are perturbations by the scaling operators.

Now the choice of the perturbing parameters is not at all unique, as we could as well have chosen [10] another convenient set of parameters S_α, and we consider the family of deformed theories characterized by

$$W(X, S_\alpha) = W_0 + \sum_{\alpha=1}^{9} S_\alpha X_\alpha.$$

We can therefore trade the couplings t_α for any other alternative choice. In particular a very convenient choice would be the so called *flat* or *special* coordinates in the space of deformations of the theory. Thus we may choose the elements as the set S_α to be expressible in the form

$$S_\alpha = g_\alpha(t)$$

where the g_α are unknown functions to be determined and t is the dimensionless *flat* coordinate defined earlier. In this way we can consider more general deformations (i.e. those which depend not only linearly on the couplings) of the superpotential, while at the same time characterize all these deformations by a single parameter t. Imposing certain restrictions for these deformations to be 'flat' these generic couplings g_α can be completely and uniquely determined. We also note that, in accordance with our interpretation of the deformations, these couplings should all satisfy

$$g_\alpha(t) \sim t_\alpha + \mathcal{O}(t^2).$$

Assuming the existence of a system of *flat* coordinates, the deformation of the superpotential in terms of the *flat* coordinates can be generically written as

$$W(X,t) = W_0(x,y) + \sum_{\alpha=1}^{9} g_\alpha(t) X_\alpha$$

where the suitable forms of the couplings g_α are to be determined. Explicitly, the form of the perturbed superpotential is given by

$$W(X,t) = \tfrac{1}{4}x^4 + \tfrac{1}{4}y^4 + g_1(t)1 + g_2(t)x + g_3(t)y + g_4(t)x^2 + g_5(t)xy + g_6(t)y^2 \\ + g_7(t)x^2y + g_8(t)xy^2 + g_9(t)x^2y^2.$$

According to the formula the perturbed fields which still satisfy the ring structure (though the ring now is no longer nilpotent) are given by

$$\phi_\alpha(X,t) = \frac{\partial W(X,t)}{\partial t_\alpha} = \sum_\beta \frac{\partial g_\beta(t)}{\partial t_\alpha} X_\beta.$$

Once again apart from the coupling $g_9(t)$ all the couplings have dimensions. In order to segregate this dimensional dependence, we introduce a dimensional parameter s, with $U(1)$ dimension $\frac{1}{4}$ to pull out the dimensional dependence from the couplings and redefine the couplings in terms of dimensionless parameters $\alpha(t)$ ($= g_9(t)$) and $\beta_i(t)$ for $i = 1, 2, \ldots 8$, as follows

$$W(x,y;t,s) = \frac{1}{4}x^4 + \frac{1}{4}y^4 + \frac{1}{2}\alpha(t)x^2y^2 + s\beta_1(t)x^2y + s\beta_2(t)xy^2 + \frac{1}{2!}s^2\beta_3(t)x^2 + \frac{1}{2!}s^2\beta_4(t)y^2 \\ + \frac{1}{2!}s^2\beta_5(t)xy + \frac{1}{3!}s^3\beta_6(t)x + \frac{1}{3!}s^3\beta_7(t)y + \frac{1}{4!}s^4\beta_8(t)1. \quad (1.4)$$

The unique chiral primary field with the maximal $U(1)$ charge is now given by

$$\phi_9(X,t) = \frac{\partial W(X,t)}{\partial t_9} = \frac{\partial W(X,t)}{\partial g_9(t)} \frac{\partial g_9(t)}{\partial t} = x^2y^2 g_9'(t) = \alpha'(t)x^2y^2. \quad (1.5)$$

Note that this reduces to simply x^2y^2 in the free theory. Let us now consider the theory with only the marginal perturbation, by setting $s = 0$. The theory is then defined by the superpotential

$$W(X,t) = \tfrac{1}{4}x^4 + \tfrac{1}{4}y^4 + \tfrac{1}{2}\alpha(t)x^2y^2. \tag{1.6}$$

The Hessian of the above superpotential at the criticality is given by

$$h_0(x,y) \sim 9(1-\alpha^2){x_0}^2{y_0}^2$$

(where the subscript 0 refers to the values of the quantities at the critical points of W), and is clearly non-degenerate for $\alpha^2 \neq 1$.

The Grothendieck metric [14], η of the model is given in this case by the expectation value of the field with the maximal $U(1)$ charge (1.5) (note that the superpotential defined above is still quasihomogeneous and hence $U(1)$ charge conservation has to be satisfied), and can be calculated by the prescription due to Vafa (see [15]) as follows

$$\begin{aligned}\eta \equiv \langle\phi_9(X,t)\rangle &= \alpha'(t)\langle x^2y^2\rangle \\ &= \alpha'(t)\oint_{C_x}\oint_{C_y}\frac{\mathrm{d}x}{2\pi\mathrm{i}}\frac{\mathrm{d}y}{2\pi\mathrm{i}}\frac{x^2y^2}{(\partial_x W)(\partial_y W)}\end{aligned}$$

(where the contours C_x and C_y are large enough to contain all the zeroes of ∂W)

$$\begin{aligned}&= \alpha'(t)\oint_{C_x}\oint_{C_y}\frac{\mathrm{d}x}{2\pi\mathrm{i}}\frac{\mathrm{d}y}{2\pi\mathrm{i}}\frac{x^2y^2}{(x^3+\alpha xy^2)(y^3+\alpha x^2y)} \\ &= \alpha'(t)\oint_{C_x}\oint_{C_y}\frac{\mathrm{d}x}{2\pi\mathrm{i}}\frac{\mathrm{d}y}{2\pi\mathrm{i}}\frac{1}{xy}\left(1+\alpha\frac{y^2}{x^2}\right)^{-1}\left(1+\alpha\frac{x^2}{y^2}\right)^{-1} \\ &= \alpha'(t)[1+\alpha^2+\alpha^4+\alpha^6+\cdots] \\ &= \left(\frac{\alpha'(t)}{1-\alpha^2(t)}\right).\end{aligned} \tag{1.7}$$

The value of the Grothendieck metric would have been simply 1 (according to our normalizations) in the absence of any perturbations†.

The action for a general $N = 2$ SCFT is given by

$$S = \int \mathrm{d}^2z\,\mathrm{d}^2\theta\,K(X_i,\overline{X}_i) + \left(\int \mathrm{d}^2z\,\mathrm{d}^2\theta\,W(X_i) + \mathrm{CC}\right)$$

where W is a holomorphic function of the chiral superfields X_i. The renormalization-group (RG) flow for such a theory is driven solely by the superpotential W (the kinetic term providing only irrelevant perturbations). In fact, W is presumably not renormalized—and hence provides us with an invariant of the RG flow to characterize such two-dimensional theories. We shall use this information when we discuss the path integral for such a model.

† The solution to the equation $\alpha'(t)/(1-\alpha^2(t)) = 1$ is given by $\alpha(t) = (\alpha(0)+\tanh t)/(1+\alpha(0)\tanh t)$.

Let us now consider the superpotential $\widetilde{W}$ defined by

$$\widetilde{W}(x, y, w) = W_0(x, y) + \tfrac{1}{2}w^2 = \tfrac{1}{4}x^4 + \tfrac{1}{4}y^4 + \tfrac{1}{2}\alpha(t)x^2y^2 + \tfrac{1}{2}w^2 \tag{1.8}$$

where $w(z, \overline{z})$ is a Landau–Ginzburg field of dimension $\frac{1}{2}$. Note that the addition does not alter the central charge of the theory. In order to discuss some algebraic and geometric aspects of the above model, it is convenient to use a slightly different normalization of the fields. Accordingly we define

$$X_1 = \frac{x}{\sqrt{2}} \qquad X_2 = \frac{y}{\sqrt{2}} \qquad X_3 = \frac{w}{\sqrt{2}} \qquad a(t) = 2\alpha(t).$$

The superpotential $\widetilde{W}$ now looks like

$$\widetilde{W}(X_1, X_2, X_3) = X_1{}^4 + X_2{}^4 + X_3{}^2 + a(t)X_1{}^2X_2{}^2.$$

We can now introduce a system of projective coordinates (ξ_1, ξ_2, ξ_3) for parametrizing the superpotential $\tilde{W}$ as follows

$$\xi_1 = X_1{}^4 \qquad \xi_2 = \frac{X_2}{X_1} \qquad \xi_3 = \frac{X_3}{X_1{}^2}.$$

It may be noted that the coordinate ξ_1 has dimension 1, while the coordinates ξ_2 and ξ_3 are dimensionless.

The Jacobian $J = |\partial(X_1, X_2, X_3)/\partial(\xi_1, \xi_2, \xi_3)|$ for the above transformation turns out to be a purely numerical factor. In terms of the homogeneous coordinates the superpotential becomes

$$\widetilde{W}(\xi_1, \xi_2, \xi_3) = \xi_1[1 + \xi_2{}^4 + \xi_3{}^2 + a(t)\xi_2{}^2].$$

The path integral for the model defined by the Landau–Ginzburg superpotential reduces to:

$$\int [\mathrm{d}^2X_1][\mathrm{d}^2X_2][\mathrm{d}^2X_3]\exp\left(\mathrm{i}\int \mathrm{d}^2z\,\mathrm{d}^2\theta\,\tilde{W}(X_1, X_2, X_3)\right)$$
$$\longrightarrow \int [\mathrm{d}^2\xi_1][\mathrm{d}^2\xi_2][\mathrm{d}^2\xi_3]|J|^2\exp\left(\mathrm{i}\int \mathrm{d}^2z\,\mathrm{d}^2\theta\,\xi_1[1 + \xi_2{}^4 + \xi_3{}^2 + a(t)\xi_2{}^2]\right).$$

Performing the trivial integral over the ξ_1 then leaves us with the delta functional

$$\int [\mathrm{d}^2\xi_2][\mathrm{d}^2\xi_3]\delta[1 + \xi_2{}^4 + \xi_3{}^2 + a(t)\xi_2{}^2]$$

In other words, we get a path integral over a complex one-dimensional hypersurface, on the vanishing set of $\tilde{W}$ evaluated in the projective coordinates $(1, \xi_2, \xi_3)$. The delta function constraint imposes the equation

$$1 + \xi_2{}^4 + \xi_3{}^2 + a(t)\xi_2{}^2 = 0$$

or equivalently

$$\widetilde{W}(X_1, X_2, X_3) = \Sigma = X_1{}^4 + X_2{}^4 + X_3{}^2 + a(t)X_1{}^2X_2{}^2 = 0. \tag{1.9}$$

Setting $\widetilde{W} = 0$ in projective 2-space, we get a one-dimensional torus, whose moduli are fixed by a. Further by virtue of the existence of a quantum $\mathbb{Z}_4$ symmetry of $\widetilde{W}$ the volume of the torus is also fixed to be 1 for all a. The coordinates (ξ_2, ξ_3) can be regarded as a patch on a two-dimensional *weighted projective space* $\mathbb{WCP}_2$ in which

$$(X_1, X_2, X_3) \simeq (e^{2\pi i/4}X_1, e^{2\pi i/4}X_2, e^{2\pi i/2}X_3). \tag{1.10}$$

The curve Σ defined in the *weighted projective space* $\mathbb{WCP}_2$ by equation (1.9) is obtained by the identification (1.10). The patch obviously excludes the point $X_1 = 0$. The patch selected is, however, not unique as other patches can be obtained by coordinate transformations. Curves such as the one Σ described above have been widely studied and classified. The curve defined by Σ has signature $(g; e) = (1, \infty, \infty)$, where g lists the genus and e_i, the orders of the branch points of the curve in the *weighted projective space*. Σ is essentially an orbifold of tori $\mathbb{C}/\tilde{\Gamma}$, where $\tilde{\Gamma}$ is a discrete subgroup $ISO(2)$ consisting of lattice translations together with $SO(2)$ rotations by angles $2\pi/4$. The equation $\widetilde{W} = 0$ is the relation among the ring of regular functions defined on a line bundle of degree $(-2)\widehat{E}_7$ over a torus. All this is summarized by the statement that singularity theorists classify the superpotential $\widetilde{W}(X_1, X_2, X_3)$, (with the non-degeneracy condition $a^2 \neq 4$) as belonging to the modality-1 singularity type E_7 or X_9 type [6]; the Landau–Ginsburg superpotential

$$\widetilde{W}(X_1, X_2,) = X_1{}^4 + X_2{}^4 + a(t)X_1{}^2X_2{}^2$$

being equivalent to the orbifold $SO(4)/\mathbb{Z}_4$.

Finally we make an important observation: the physics described by the superpotential $W(x, y)$ and the superpotential $\widetilde{W}(x, y, w) = W(x, y) + \frac{1}{2}w^2$, where the superpotential $W(x, y)$ is given by equation (1.6) or by equation (1.4), are *exactly identical*. This is because the Landau–Ginzburg field w appears only quadratically in the action and, hence, may be readily integrated out in the path integral to give us merely a constant phase factor.

2. Explicit solution of the problem

In order to obtain the *flat* coordintes characterizing the most general deformation of the superpotential, we follow the method of Lerche *et al* (see [7]) which, in our present case, essentially reduces to, first constructing the function:

$$U(t, s) = (-1)^\lambda \Gamma(\lambda) \int [dx] \frac{q(t)}{[W^\lambda(t, s)]}$$

where t is the dimensionless *flat* coordinate, and s is a parameter of dimension $\frac{1}{4}$ which pulls out the dimensional dependence from the other flat coordinates. In general, it can be shown (for details see [7]) that the following equation holds:

$$\frac{\partial^2}{\partial s_i \partial s_j} U = C_{ij}{}^\alpha (-1)^{\lambda+2} \Gamma(\lambda + 2) \int [dx] \frac{\chi_\alpha}{[W^{\lambda+2}(t, s)]} + \Gamma_{ij}{}^k \left(\frac{\partial}{\partial s_k}\right) U \tag{2.1}$$

where s_i parametrizes a general, versal deformation of the superpotential; our problem is now reduced to now finding the relation between these general coordinates s_i and the flat coordinates t_i. (χ_α, above is a polynomial basis for the chiral ring.) The $C_{ij}{}^k$ are the same as the structure constants of the ring, while Γ is essentially the Gauss–Mannin connection. Flat coordinates are determined by the requirement that $\Gamma \equiv 0$ in the above equation.

2.1. Determination of the marginal coupling α

For the first part of our calculation, where we wish to solve for the coupling of the marginal perturbation, it suffices to concentrate only on the $s = 0$ piece of our perturbed superpotential. Accordingly, we construct our function $U(t, 0)$ as

$$U(t,0) = (-1)^{\lambda}\Gamma(\lambda)\int [\mathrm{d}x]\frac{q(t)}{W^{\lambda}}$$

where, as usual, $[\mathrm{d}x] = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n$ and $W = W(x, y, t, s = 0)$ is the perturbed superpotential, perturbed by the marginal perturbation, and is explicitly given by

$$W = \tfrac{1}{4}x^4 + \tfrac{1}{4}y^4 + \tfrac{1}{2}\alpha(t)x^2y^2 \tag{2.2}$$

where $\alpha^2 \neq 1$ from the criterion of non-degeneracy of W. We want to solve for the coupling $\alpha(t)$ by requiring the connection Γ to be flat. From the above definition, we readily obtain the equation

$$\frac{\partial^2}{\partial t^2}U(t,0) = \frac{q''}{q}U(t,0) + (-1)^{\lambda+1}\Gamma(\lambda+1)\int [\mathrm{d}x]\frac{1}{W^{\lambda+1}}\left(qW'' + 2q'W'\right) + (-1)^{\lambda+2}\Gamma(\lambda+2)\int [\mathrm{d}x]\frac{1}{W^{\lambda+2}}q(W')^2$$

where all primes refer to derivatives with respect to the dimensionless parameter t. The flatness condition then essentially reduces to demanding the vanishing of the terms proportional to $\sim 1/W^{\lambda+1}$ and those proportional to $U(t, 0)$. Using the explicit form of the perturbed superpotential (2.2), the above equation reduces to

$$\frac{\partial^2}{\partial t^2}U(t,0) = \frac{q''}{q}U(t,0) + (-1)^{\lambda+1}\Gamma(\lambda+1)\int [\mathrm{d}x]\frac{1}{W^{\lambda+1}}\left(\frac{\alpha''}{\alpha'} + 2\frac{q'}{q}\right)\frac{1}{2}q\alpha' x^2y^2 + (-1)^{\lambda+2}\Gamma(\lambda+2)\int [\mathrm{d}x]\frac{1}{W^{\lambda+2}}\left(\frac{1}{4}q\alpha'^2\right)x^4y^4. \tag{2.3}$$

Integrating the last term successively by parts to reduce its degree, we obtain, after a straightforward but rather lengthy calculation, the following simple result:

$$\frac{\partial^2}{\partial t^2}U(t,0) = \left[\frac{q''}{q} + \frac{1}{4}\left(\frac{\alpha'^2}{1-\alpha^2}\right)\right]U(t,0) + (-1)^{\lambda+1}\Gamma(\lambda+1) \times \int [\mathrm{d}x]\frac{1}{W^{\lambda+1}}\left[\frac{\alpha''}{\alpha'} + 2\frac{q'}{q} + 2\left(\frac{\alpha\alpha'}{1-\alpha^2}\right)\right]\frac{1}{2}q\alpha' x^2y^2. \tag{2.4}$$

In obtaining the above result we have utilized the following identities in our simplifications

$$\text{(i) } x^4y^4 = \frac{1}{1-\alpha^2}\left[x^4y\frac{\partial W}{\partial y} + y^4x\frac{\partial W}{\partial x} - xy\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}\right] \tag{2.5}$$

$$\text{(ii) } x^4 + y^4 = \left[x\frac{\partial W}{\partial x} + y\frac{\partial W}{\partial y} - 2\alpha x^2y^2\right]. \tag{2.6}$$

We have also used the result

$$0 = \int [\mathrm{d}x] \frac{\partial}{\partial x^A}\left(\frac{V^A}{W^{\lambda+1}}\right)$$

for any vector V^A, to throw away the surface terms.

Now from the above equation, demanding that the terms proportional to $\sim 1/W^{\lambda+1}$, and those proportinal to $U(t, 0)$ should vanish separately, we obtain the following equations:

$$\frac{\alpha''}{\alpha'} + 2\frac{q'}{q} + 2\left(\frac{\alpha\alpha'}{1-\alpha^2}\right) = 0 \tag{2.7}$$

$$\frac{q''}{q} + \frac{1}{4}\left(\frac{\alpha'^2}{1-\alpha^2}\right) = 0. \tag{2.8}$$

The first of the above equations can be readily integrated to give

$$q(t) = \left(\frac{1-\alpha^2(t)}{\alpha'(t)}\right)^{1/2}. \tag{2.9}$$

We can here observe that the factor $\left[q^{-2}(t)\right]$, is essentially the conformal rescaling factor that takes the Grothendieck metric (see equation (1.7)), of $W(x, y, t)$ to the flat metric.

Using the above solution to eliminate $q(t)$ between the above equations, we finally obtain the following Schwarzian differential equation for the quantity $\alpha(t)$:

$$\{\alpha; t\} = -\frac{1}{2}\frac{3+\alpha^2}{(1-\alpha^2)^2}(\alpha'^2) \tag{2.10}$$

where the Schwarzian derivative $\{;\}$ is defined by

$$\{\alpha; t\} = \frac{\alpha'''}{\alpha'} - \frac{3}{2}\left(\frac{\alpha''}{\alpha'}\right)^2$$

(where $\alpha' = \mathrm{d}\alpha/\mathrm{d}t$ etc). We therefore need to solve the above equation (2.10) to obtain the expression for $\alpha(t)$. We note that if we trade the parameter $\alpha(t)$ for a new variable $\theta(t)$, defined by

$$\alpha(t) = \sin\theta(t) \tag{2.11}$$

the above equation may be reduced to the form:

$$\{\theta; t\} = -\tfrac{1}{2}\theta'^2\sec^2\theta \qquad \text{where } \theta' = \mathrm{d}\theta/\mathrm{d}t \qquad \text{and} \qquad \theta = \theta(t).$$

Further, using the properties of Schwarzian derivatives, the above may be written in the form:

$$\{t; \theta\} = \tfrac{1}{2}\sec^2\theta \qquad \text{where } t = t(\theta) \qquad \text{and} \qquad t' = \mathrm{d}t/\mathrm{d}\theta \qquad \text{etc.} \tag{2.12}$$

The above equation (2.12) may be readily integrated once to give

$$\Psi(\theta) = \Psi(0) + \frac{1}{2}e^{\theta/2}\int_0^\theta dx\, e^{-x/2}\sec^2 x$$

where $\Psi(\theta) = t''(\theta)/t'(\theta)$. Finally integrating twice, we can obtain the following solution for $t(\theta)$:

$$t(\theta) = t(0) + t'(0)\int_0^\theta dz\, e^{m(z)} \tag{2.13}$$

where the function m(z) is defined by

$$m(z) = z\left(\frac{t''(0)}{t'(0)}\right) + \frac{1}{2}\int_0^z dy \int_0^y dx\, e^{-(x-y)/2}\sec^2 x. \tag{2.14}$$

We note that the integrals are not reducible any further in terms of any known elementary functions. The task remaining is, therefore, to invert the above solution to obtain $\theta(t)$ and then to finally solve for $\alpha(t)$ using the relation (2.11).

Alternatively, in order to solve the above equation, we note that using the well known properties of Schwarzian derivatives, we can rewrite the equation (2.10) in the form

$$\{t;\alpha\} = \frac{1}{2}\frac{3+\alpha^2}{\left(1-\alpha^2\right)^2}. \tag{2.15}$$

We also observe, in passing that using a further change of variable to $x = \alpha^{-2}$ we can reduce the above equation to the form

$$\{t;x\} = \frac{3}{8x^2} + \frac{3}{8}\frac{1}{x(x-1)^2} + \frac{1}{8}\frac{1}{x^2(x-1)^2} \tag{2.16}$$

(where $t = t(x)$). Further we can make an important technical observation here. If we trade the function $\alpha(t)$, for the function $\gamma(t)$ defined by

$$\gamma(t) = \frac{\left(\alpha^2+3\right)^3}{27\left(1-\alpha^2\right)^2} \tag{2.17}$$

then it can be seen that the Schwarzian equation for $\{t;\gamma\}$ tells us that γ is given by the modular function, i.e.

$$\gamma(t) = J(\tau) \tag{2.18}$$

where $\tau = (at+b)/(ct+d)$ for some $a, b, c, d \in \mathbb{C}$, and $ad - bc \neq 0$ and $J = j/1728$ is the absolute modular invariant function whose q-expansion is well known to be given by $\frac{1}{1728}(\frac{1}{q} + 744 + \cdots)$. Consequently, the definition (2.17) can be inverted and the solution for $\alpha(t)$ can be written down in terms of the Schwarzian triangle functions [16].

However, we shall work with the previous form of the Schwarzian equation (2.15). It can be seen that the general solution of equation (2.15) for $t(\alpha)$ can be expressed as the ratio of two hypergeometric functions.

We know that a second-order differential equation of the form

$$\frac{d^2w}{d\mu^2} + p(\mu)\frac{dw}{d\mu} + q(\mu)w = 0 \qquad (2.19)$$

where $w = w(\mu)$ gives rise to the non-linear Schwarzian differential equation

$$\{\xi; \mu\} = 2q(\mu) - 1/2p^2(\mu) - dp/d\mu \qquad (2.20)$$

where $\xi(\mu) = w_1(\mu)/w_2(\mu)$ and $w_{1,2}$ are the two linearly independent solutions of the equation (2.19). Using the above wisdom, it is easy to see that the general solution $t(\alpha)$ of equation (2.15) can be expressed as the ratio

$$t(\alpha) = v_1(\alpha)/v_2(\alpha) \qquad (2.21)$$

where $v_i(\alpha)$, $i = 1, 2$ are the linearly independent solutions of the equation:

$$\left[(1-\alpha^2)\frac{d^2}{d\alpha^2} - 2\alpha\frac{d}{d\alpha} - \frac{1}{4}\right]v_i(\alpha) = 0. \qquad (2.22)$$

By a convenient change of variable to $z = \frac{1}{2}(1+\alpha)$, the above equation can be recast into a readily recognizable form of a hypergeometric equation:

$$\left[z(1-z)\frac{d^2}{dz^2} - (2z-1)\frac{d}{dz} - \frac{1}{4}\right]v_i(z) = 0 \qquad (2.23)$$

whose solutions are given by the hypergeometric function $F(\frac{1}{2}, \frac{1}{2}; 1, z)$.

2.2. *Determination of the relevant couplings β_i*

Next we proceed to systematically determine the couplings $\beta_i(t)$, for $i = 1, 2, \ldots, 8$. To solve for the couplings we must consider the complete form of the perturbed superpotential. Thus our function $U(t, s)$ will now be given by

$$U(t,s) = (-1)^\lambda \Gamma(\lambda) \int [dx] \frac{q(t)}{[W^\lambda(t,s)]}$$

where the perturbed superpotential is given by

$$W(x, y; t, s) = \frac{1}{4}x^4 + \frac{1}{4}y^4 + \frac{1}{2}\alpha(t)x^2y^2 + s\beta_1(t)x^2y + s\beta_2(t)xy^2 + \frac{1}{2!}s^2\beta_3(t)x^2 + \frac{1}{2!}s^2\beta_4(t)y^2 + \frac{1}{2!}s^2\beta_5(t)xy + \frac{1}{3!}s^3\beta_6(t)x + \frac{1}{3!}s^3\beta_7(t)y + \frac{1}{4!}s^4\beta_8(t)1. \qquad (2.24)$$

To solve for the couplings $\beta_1(t)$ and $\beta_2(t)$, we consider the $s = 0$ bit of the expression for $\partial^2 U/\partial t\partial s$, which is given by

$$\left(\frac{\partial^2 U(t,s)}{\partial t\partial s}\right)_{s=0} = (-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\frac{q}{W^{\lambda+2}}\left[\left(\frac{\partial W}{\partial t}\right)\left(\frac{\partial W}{\partial s}\right)\right]_{s=0}$$
$$+(-1)^{\lambda+1}\Gamma(\lambda+1)\int[\mathrm{d}x]\frac{1}{W^{\lambda+1}}\left[q'\left(\frac{\partial W}{\partial s}\right)+q\left(\frac{\partial^2 W}{\partial t\partial s}\right)\right]_{s=0}$$
$$=(-1)^{\lambda+1}\Gamma(\lambda+1)\int[\mathrm{d}x]\frac{1}{W^{\lambda+1}}(q'\beta_1+q\beta_1')x^2y$$
$$+(-1)^{\lambda+1}\Gamma(\lambda+1)\int[\mathrm{d}x]\frac{1}{W^{\lambda+1}}(q'\beta_2+q\beta_2')xy^2$$
$$+(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\frac{1}{W^{\lambda+2}}(\beta_1x+\beta_2y)\frac{1}{2}q\alpha'x^3y^3.$$

Once again using the identity (2.5) to integrate by parts the last term, we obtain, after some elaborate simplifications,

$$\left(\frac{\partial^2 U(t,s)}{\partial t\partial s}\right)_{s=0} = (-1)^{\lambda+1}\Gamma(\lambda+1)\int\frac{[\mathrm{d}x]}{W^{\lambda+1}}\left[q'\beta_1+q\beta_1{}'+\frac{3}{2}\frac{q\beta_1\alpha\alpha'}{(1-\alpha^2)}\right]x^2y$$
$$+(-1)^{\lambda+1}\Gamma(\lambda+1)\int\frac{[\mathrm{d}x]}{W^{\lambda+1}}\left[q'\beta_2+q\beta_2{}'+\frac{3}{2}\frac{q\beta_2\alpha\alpha'}{(1-\alpha^2)}\right]xy^2.$$

The conditions determining the couplings $\beta_1(t)$ and $\beta_2(t)$, therefore require

$$q'\beta_1+q\beta_1{}'+\frac{3}{2}\frac{q\beta_1\alpha\alpha'}{(1-\alpha^2)}=0 \tag{2.25}$$

$$q'\beta_2+q\beta_2{}'+\frac{3}{2}\frac{q\beta_2\alpha\alpha'}{(1-\alpha^2)}=0. \tag{2.26}$$

On substituting the value of $q(t)$ from equation (2.9), the above equations may be readily solved to give the couplings:

$$\beta_1(t)=C_1[\alpha'^2(1-\alpha^2)]^{1/4} \tag{2.27}$$

$$\beta_2(t)=C_2[\alpha'^2(1-\alpha^2)]^{1/4} \tag{2.28}$$

where C_1 and C_2 are constants.

In order to determine the couplings $\beta_3(t)$, $\beta_4(t)$ and $\beta_5(t)$, we have to proceed almost exactly in the same way, except that now we have to consider the $s = 0$ piece of $\partial^2 U/\partial s^2$. Carrying out the explicit calculations we get

$$\left(\frac{\partial^2 U}{\partial s^2}\right)_{s=0} = (-1)^{\lambda+1}\Gamma(\lambda+1)\int[\mathrm{d}x]\frac{q}{W^{\lambda+1}}\left(\frac{\partial^2 W}{\partial s^2}\right)_{s=0}$$
$$+(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\frac{q}{W^{\lambda+2}}\left(\frac{\partial W}{\partial s}\right)^2_{s=0}$$
$$=(-1)^{\lambda+1}\Gamma(\lambda+1)\int[\mathrm{d}x]\frac{q}{W^{\lambda+1}}(\beta_3x^2+\beta_4y^2+\beta_5xy)$$
$$+(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\frac{q}{W^{\lambda+2}}(\beta_1{}^2x^4y^2+\beta_2{}^2x^2y^4+2\beta_1\beta_2x^3y^3).$$

Integrating the last term by parts we finally obtain and after setting $s = 0$, throughout and collecting the similar terms, we get

$$\left(\frac{\partial^2 U}{\partial s^2}\right)_{s=0} = (-1)^{\lambda+1}\Gamma(\lambda+1)\int [\mathrm{d}x]\frac{q}{W^{\lambda+1}}\left[\beta_3 + \left(\frac{\alpha\beta_1{}^2 - \beta_2{}^2}{1-\alpha^2}\right)\right]x^2$$
$$+ (-1)^{\lambda+1}\Gamma(\lambda+1)\int [\mathrm{d}x]\frac{q}{W^{\lambda+1}}\left[\beta_4 + \left(\frac{\alpha\beta_2{}^2 - \beta_1{}^2}{1-\alpha^2}\right)\right]y^2$$
$$+ (-1)^{\lambda+1}\Gamma(\lambda+1)\int [\mathrm{d}x]\frac{q}{W^{\lambda+1}}\left[\beta_5 + \left(\frac{4\alpha\beta_1\beta_2}{1-\alpha^2}\right)\right]xy \qquad (2.29)$$

where we have made use of the identities

$$\text{(i) } x^4y^2 = (1-\alpha^2)^{-1}\left[xy^2\frac{\partial W}{\partial x} - \alpha x^2 y\frac{\partial W}{\partial y}\right] \qquad (2.30)$$

$$\text{(ii) } x^2y^4 = (1-\alpha^2)^{-1}\left[x^2y\frac{\partial W}{\partial y} - \alpha xy^2\frac{\partial W}{\partial x}\right]$$

and some of the other previously stated identities to simplify some of the intermediate steps.

The couplings are therefore determined by the equations:

$$\beta_3 + \left(\frac{\alpha\beta_1{}^2 - \beta_2{}^2}{1-\alpha^2}\right) = 0 \qquad (2.32)$$

$$\beta_4 + \left(\frac{\alpha\beta_2{}^2 - \beta_1{}^2}{1-\alpha^2}\right) = 0 \qquad (2.33)$$

$$\beta_5 + \left(\frac{4\alpha\beta_1\beta_2}{1-\alpha^2}\right) = 0. \qquad (2.34)$$

On substituting the values of the couplings $\beta_1(t)$ and $\beta_2(t)$ already obtained (see equation (2.27), (2.28)), we may readily solve the above three equations to give us the following values of the three couplings:

$$\beta_3(t) = (\mathcal{C}_2{}^2 - \alpha\mathcal{C}_1{}^2)[\alpha'(1-\alpha^2)^{-1/2}] \qquad (2.35)$$

$$\beta_4(t) = (\mathcal{C}_1{}^2 - \alpha\mathcal{C}_2{}^2)[\alpha'(1-\alpha^2)^{-1/2}] \qquad (2.36)$$

$$\beta_5(t) = -4\mathcal{C}_1\mathcal{C}_2[\alpha\alpha'(1-\alpha^2)^{-1/2}]. \qquad (2.37)$$

Finally, we come to the question of determining the remaining couplings $\beta_6(t)$, $\beta_7(t)$ and $\beta_8(t)$. Here, the calculations become much more tedious as we have to consider the pieces linear and quadratic in s in the expression for $\partial^2U/\partial s^2$. It may be remarked that even though the terms independent of s in the expansion of $\partial^2U/\partial s^2$ would finally vanish as a consequence of the choice of our couplings $\beta_3(t)$, $\beta_4(t)$ and $\beta_5(t)$, we cannot throw away all such terms from the very beginning. This caution is needed because in calculating the couplings $\beta_1(t), \ldots \beta_5(t)$, we have made use of several identities in integrating by parts some of the terms. In all such cases, we have always thrown away all s-dependent terms.

Now, however, we have to consider all s-dependent terms which may have arisen out of partial integration in intermediate steps.

We therefore explain our calculations in some more detail in this part of our discussion. Carrying out the computations explicitly, we have

$$\frac{\partial^2}{\partial s^2}U(t,s) = (-1)^{\lambda+1}\Gamma(\lambda+1)$$

$$\times \int [\mathrm{d}x]\frac{q}{W^{\lambda+1}}[\beta_3 x^2 + \beta_4 y^2 + \beta_5 xy + s\beta_6 x + s\beta_7 y + \tfrac{1}{2}s^2\beta_8]$$

$$+ (-1)^{\lambda+2}\Gamma(\lambda+2)\int [\mathrm{d}x]\frac{q}{W^{\lambda+2}}[\beta_1 x^2 y + \beta_2 x y^2 + s\beta_3 x^2 + s\beta_4 y^2 + s\beta_5 xy$$

$$+ \tfrac{1}{2}s^2\beta_6 x + \tfrac{1}{2}s^2\beta_7 y + \tfrac{1}{6}s^3\beta_8]^2.$$

Expanding out, and keeping terms up to quadratic order in s, we have

$$\frac{\partial^2}{\partial s^2}U(t,s) = (-1)^{\lambda+1}\Gamma(\lambda+1)\int [\mathrm{d}x]\frac{q}{W^{\lambda+1}}(\beta_3 x^2 + \beta_4 y^2 + \beta_5 xy)$$

$$+ (-1)^{\lambda+2}\Gamma(\lambda+2)\int [\mathrm{d}x]\frac{q}{W^{\lambda+2}}(\beta_1 x^2 y + \beta_2 x y^2)^2$$

$$+ (-1)^{\lambda+1}\Gamma(\lambda+1)\int [\mathrm{d}x]\frac{q}{W^{\lambda+1}}s(\beta_6 x + \beta_7 y)$$

$$+ (-1)^{\lambda+2}\Gamma(\lambda+2)\int [\mathrm{d}x]\frac{2qs}{W^{\lambda+2}}[(\beta_1\beta_4 + \beta_2\beta_5)x^2y^3 + (\beta_1\beta_5 + \beta_2\beta_3)x^3y^2$$

$$+ \beta_1\beta_3 x^4 y + \beta_2\beta_4 x y^4]$$

$$+ (-1)^{\lambda+1}\Gamma(\lambda+1)\int [\mathrm{d}x]\frac{q}{W^{\lambda+1}}\frac{1}{2}s^2\beta_8$$

$$+ (-1)^{\lambda+2}\Gamma(\lambda+2\int [\mathrm{d}x]\frac{q}{W^{\lambda+2}}s^2[\beta_3{}^2x^4 + \beta_4{}^2y^4 + 2\beta_4\beta_5 x y^3 + 2\beta_3\beta_5 x^3 y$$

$$+ \beta_1\beta_6 x^3 y + \beta_2\beta_7 x y^3 + (\beta_1\beta_7 + \beta_2\beta_6 + \beta_5{}^2 + 2\beta_3\beta_4)x^2y^2] + \mathcal{O}(s^3)\ldots$$

$$= (-1)^{\lambda+1}\Gamma(\lambda+1)\int [\mathrm{d}x]\frac{q}{W^{\lambda+1}}(\beta_3 x^2 + \beta_4 y^2 + \beta_5 xy)$$

$$+ (-1)^{\lambda+2}\Gamma(\lambda+2)\int [\mathrm{d}x]\frac{q}{W^{\lambda+2}}(\beta_1{}^2x^4y^2 + \beta_2{}^2x^2y^4 + 2\beta_1\beta_2 x^3y^3)$$

$$+ (-1)^{\lambda+1}\Gamma(\lambda+1)\int [\mathrm{d}x]\frac{q}{W^{\lambda+1}}s(\beta_6 x + \beta_7 y)$$

$$+ (-1)^{\lambda+2}\Gamma(\lambda+2)\int [\mathrm{d}x]\frac{q}{W^{\lambda+2}}2s(\beta_1\beta_3 x^4 y + \beta_2\beta_4 x y^4)$$

$$+ (-1)^{\lambda+2}\Gamma(\lambda+2)\int [\mathrm{d}x]\frac{2qs}{W^{\lambda+2}}(\beta_1\beta_4 + \beta_2\beta_5)\frac{1}{\alpha}\bigg[xy\frac{\partial W}{\partial x}$$

$$- x^4 y - 2s\beta_1 x^2 y^2 - s\beta_2 x y^3\bigg]$$

$$+(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\frac{2qs}{W^{\lambda+2}}(\beta_1\beta_5+\beta_2\beta_3)\frac{1}{\alpha}\left[xy\frac{\partial W}{\partial y}\right.$$
$$\left.-xy^4-2s\beta_2x^2y-s\beta_1x^3y\right]$$
$$+(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\frac{qs^2}{W^{\lambda+2}}[(x^4\beta_3{}^2+y^4\beta_4{}^2)$$
$$+(\beta_5{}^2+\beta_1\beta_7+\beta_2\beta_6+2\beta_3\beta_4)x^2y^2+(\beta_2\beta_7+2\beta_4\beta_5)xy^3$$
$$+(\beta_1\beta_6+2\beta_3\beta_5)x^3y]$$

where in obtaining the above equation, we have considered terms that are at most of quadratic order in s, and have also made use of the following two identities to rewrite some of the terms in a form amenable to integration by parts:

$$\text{(i) } \alpha x^2y^3=\left[xy\frac{\partial W}{\partial x}-x^4y-2s\beta_1x^2y^2-s\beta_2xy^3+\mathcal{O}(s^2)\ldots\right] \tag{2.38}$$

$$\text{(ii) } \alpha x^3y^2=\left[xy\frac{\partial W}{\partial y}-xy^4-2s\beta_2x^2y^2-s\beta_1x^3y+\mathcal{O}(s^2)\ldots\right]. \tag{2.39}$$

After some elaborate simplifications, (some of the details are discussed in the appendix), we finally have the following result.

The piece linear in s in $\partial^2U/\partial s^2$ is given by

$$=(-1)^{\lambda+1}\Gamma(\lambda+1)\int[\mathrm{d}x]\frac{qs}{W^{\lambda+1}}x\left[\beta_6+\frac{2\alpha}{1-\alpha^2}(\beta_1\beta_5+\beta_2\beta_3)\right.$$
$$\left.-\left(\frac{2\beta_2\beta_4}{1-\alpha^2}\right)+\frac{\beta_1{}^2\beta_2(5+4\alpha^2)-\alpha\beta_2{}^3}{(1-\alpha^2)^2}\right]$$
$$+(-1)^{\lambda+1}\Gamma(\lambda+1)\int[\mathrm{d}x]\frac{qs}{W^{\lambda+1}}y\left[\beta_7+\frac{2\alpha}{1-\alpha^2}(\beta_1\beta_4+\beta_2\beta_5)\right.$$
$$\left.-\left(\frac{2\beta_1\beta_3}{1-\alpha^2}\right)+\frac{\beta_1\beta_2{}^2(5+4\alpha^2)-3\alpha\beta_1{}^3}{(1-\alpha^2)^2}\right].$$

The couplings are completely determined by the conditions:

$$\left[\beta_6+\frac{2\alpha}{1-\alpha^2}(\beta_1\beta_5+\beta_2\beta_3)-\left(\frac{2\beta_2\beta_4}{1-\alpha^2}\right)+\frac{\beta_1{}^2\beta_2(5+4\alpha^2)-3\alpha\beta_2{}^3}{(1-\alpha^2)^2}\right]=0 \tag{2.40}$$

$$\left[\beta_7+\frac{2\alpha}{1-\alpha^2}(\beta_1\beta_4+\beta_2\beta_5)-\left(\frac{2\beta_1\beta_3}{1-\alpha^2}\right)+\frac{\beta_1\beta_2{}^2(5+4\alpha^2)-3\alpha\beta_1{}^3}{(1-\alpha^2)^2}\right]=0. \tag{2.41}$$

Using the values of the couplings $\beta_1(t), \ldots, \beta_5(t)$, the above equations can therefore be solved to give us the following values of the couplings:

$$\beta_6(t)=C_2(\alpha')^{3/2}(1-\alpha^2)^{-5/4}[3C_1{}^2(2\alpha^2-1)-\alpha C_2{}^2] \tag{2.42}$$

$$\beta_7(t)=C_1(\alpha')^{3/2}(1-\alpha^2)^{-5/4}[3C_2{}^2(2\alpha^2-1)-\alpha C_1{}^2]. \tag{2.43}$$

Hence at the end of the day, we have the following complete list of all the relevant couplings—completely determined as functions of the dimensionless flat coordinate t:

$$\beta_1(t) = C_1[\alpha'^2(1-\alpha^2)]^{1/4} \tag{2.44}$$

$$\beta_2(t) = C_2[\alpha'^2(1-\alpha^2)]^{1/4} \tag{2.45}$$

$$\beta_3(t) = (C_2{}^2 - \alpha C_1{}^2)[\alpha'(1-\alpha^2)^{-1/2}] \tag{2.46}$$

$$\beta_4(t) = (C_1{}^2 - \alpha C_2{}^2)[\alpha'(1-\alpha^2)^{-1/2}] \tag{2.47}$$

$$\beta_5(t) = -4C_1C_2[\alpha\alpha'(1-\alpha^2)^{-1/2}] \tag{2.48}$$

$$\beta_6(t) = C_2(\alpha')^{3/2}(1-\alpha^2)^{-5/4}[3C_1{}^2(2\alpha^2-1) - \alpha C_2{}^2] \tag{2.49}$$

$$\beta_7(t) = C_1(\alpha')^{3/2}(1-\alpha^2)^{-5/4}[3C_2{}^2(2\alpha^2-1) - \alpha C_1{}^2]. \tag{2.50}$$

It can be verified that for the above choice of couplings, the Gauss–Manin connection Γ is completely flat. All the couplings being completely determined, the free energy $\mathcal{F}$ of the system can now be obtained; hence all the correlation functions of the perturbed as well as the unperturbed theory can be computed (since the free energy $\mathcal{F}$ acts as a generator of the correlation functions of the theory [1]). The model can thus be considered as being completely solved .

Appendix

In obtaining the expressions for the terms linear and quadratic in s in the expansion of $\partial^2 U/\partial s^2$, we have to rewrite some of the terms in a form which can be readily integrated by parts in order to reduce its degree. Thus, for example, we can rewrite one of the terms as follows

$$\begin{aligned}(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\left(\frac{q}{W^{\lambda+2}}\right)\beta_1{}^2x^4y^2 \\ = (-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\frac{q}{W^{\lambda+2}}\left(\frac{\beta_1{}^2}{1-\alpha^2}\right)\Bigg[xy^2\left(\frac{\partial W}{\partial x}\right) - \alpha x^2 y\left(\frac{\partial W}{\partial y}\right) \\ + s(\alpha\beta_1 x^4 y - \beta_2 xy) + 2s(\alpha\beta_2 x^3y^2 - \beta_1 x^2y^3) + s^2(\alpha\beta_4 - \beta_3)x^2y^2 \\ + \tfrac{1}{2}s^2\beta_5(\alpha x^3 y - xy^3)\Bigg]\end{aligned} \tag{A.1}$$

where we have made use of the identity

$$\begin{aligned}(1-\alpha^2)x^4y^2 = \Bigg[xy^2\left(\frac{\partial W}{\partial x}\right) - \alpha x^2 y\left(\frac{\partial W}{\partial y}\right) + s(\alpha\beta_1 x^4 y - \beta_2 xy) \\ + 2s(\alpha\beta_2 x^3y^2 - \beta_1 x^2y^3) + s^2(\alpha\beta_4 - \beta_3)x^2y^2 + \tfrac{1}{2}s^2\beta_5(\alpha x^3 y - xy^3)\Bigg].\end{aligned} \tag{A.2}$$

The term $\sim x^2y^4/W^{\lambda+2}$ can also be similarly reduced. Then again, using the identity

$$(1-\alpha^2)x^3y^3 = \left[x^3\left(\frac{\partial W}{\partial y}\right) + y^3\left(\frac{\partial W}{\partial x}\right) - \left(\frac{\partial W}{\partial x}\right)\left(\frac{\partial W}{\partial y}\right) + 3s\alpha(\beta_1x^3y^2+\beta_2x^2y^3)\right.$$
$$+ s^2\alpha(\beta_4+2\beta_2{}^2)xy^3 + s^2(\alpha\beta_3+2\beta_1{}^2)x^3y$$
$$\left. + s^2(\alpha\beta_5+5\beta_1\beta_2)x^2y^2 + \mathcal{O}(s^3)\ldots\right] \tag{A.3}$$

we can reduce one of the other expressions in the following way:

$$(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\frac{1}{W^{\lambda+2}}2q\beta_1\beta_2x^3y^3$$
$$= (-1)^{\lambda+1}\Gamma(\lambda+1)\int[\mathrm{d}x]\left(\frac{2q\beta_1\beta_2}{1-\alpha^2}\right)\frac{1}{W^{\lambda+1}}\left[2\alpha xy + 2s(\beta_1x+\beta_2y) + \frac{1}{2}s^2\beta_5\right]$$
$$\times(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\left(\frac{2q\beta_1\beta_2}{W^{\lambda+2}}\right)\frac{1}{1-\alpha^2}[3s\alpha(\beta_1x^3y^2+\beta_2x^2y^3)$$
$$+ s^2\alpha(\beta_4+2\beta_2{}^2)xy^3 + s^2(\alpha\beta_3+2\beta_1{}^2)x^3y$$
$$+ s^2(\alpha\beta_5+5\beta_1\beta_2)x^2y^2 + \mathcal{O}(s^3)\ldots]. \tag{A.4}$$

Adding up all the contributions, and keeping all terms to quadratic order in s, we see that the terms independent of s, in fact, cancel out by virtue of our choice of the couplings $\beta_3, \beta_4, \beta_5$. Integrating by parts the remaining terms, we then have

$$\frac{\partial^2U}{\partial s^2} = (-1)^{\lambda+1}\Gamma(\lambda+1)\int[\mathrm{d}x]\left(\frac{qs}{W^{\lambda+1}}\right)\left[(x\beta_6+y\beta_7) + \left(\frac{4\beta_1\beta_2}{1-\alpha^2}\right)(\beta_1x+\beta_2y)\right]$$
$$-(-1)^{\lambda+1}\Gamma(\lambda+1)\int[\mathrm{d}x]\left(\frac{qs}{W^{\lambda+1}}\right)\frac{2}{\alpha}[x(\beta_1\beta_5+\beta_2\beta_3)+y(\beta_1\beta_4+\beta_2\beta_5)]$$
$$+(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\left(\frac{qs}{W^{\lambda+2}}\right)x^4y$$
$$\times\left[2\beta_1\beta_3 - \frac{2}{\alpha}(\beta_1\beta_4+\beta_2\beta_5) + \left(\frac{\alpha\beta_1{}^3-\beta_1\beta_2{}^2}{1-\alpha^2}\right)\right]$$
$$+(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\left(\frac{qs}{W^{\lambda+2}}\right)xy^4$$
$$\times\left[2\beta_2\beta_4 - \frac{2}{\alpha}(\beta_1\beta_5+\beta_2\beta_3) + \left(\frac{\alpha\beta_2{}^3-\beta_2\beta_1{}^2}{1-\alpha^2}\right)\right]$$
$$+(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\left(\frac{qs}{W^{\lambda+2}}\right)\frac{x^3y^2}{1-\alpha^2}[2\alpha\beta_2\beta_1{}^2 - 2\beta_2{}^3 + 6\alpha\beta_1{}^2\beta_2]$$
$$+(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\left(\frac{qs}{W^{\lambda+2}}\right)\frac{x^2y^3}{1-\alpha^2}[2\alpha\beta_1\beta_2{}^2 - 2\beta_1{}^3 + 6\alpha\beta_2{}^2\beta_1]$$
$$+(-1)^{\lambda+2}\Gamma(\lambda+2)\int[\mathrm{d}x]\left(\frac{qs^2}{W^{\lambda+2}}\right)(x^4\beta_3{}^2 + y^4\beta_4{}^2)$$

$$+ (-1)^{\lambda+1}\Gamma(\lambda+1)\int [\mathrm{d}x]\frac{1}{W^{\lambda+1}}\left[qs^2\left(\frac{\beta_1\beta_2}{1-\alpha^2}\right)\beta_5+\frac{1}{2}\beta_8\right]$$
$$+ (-1)^{\lambda+2}\Gamma(\lambda+2)\int [\mathrm{d}x]\frac{qs^2x^2y^2}{W^{\lambda+2}}\left[\beta_1{}^2\left(\frac{\alpha\beta_4-\beta_3}{1-\alpha^2}\right)+\beta_2{}^2\left(\frac{\alpha\beta_3-\beta_4}{1-\alpha^2}\right)\right.$$
$$+\left(\frac{2\beta_1\beta_2}{1-\alpha^2}\right)(\alpha\beta_5+5\beta_1\beta_2)+\left(\beta_5{}^2+\beta_1\beta_7+\beta_2\beta_6+2\beta_3\beta_4\right)$$
$$\left.-\frac{4\beta_1}{\alpha}(\beta_1\beta_4+\beta_2\beta_5)-\frac{4\beta_2}{\alpha}(\beta_1\beta_5+\beta_2\beta_3)\right]. \tag{A.5}$$

In obtaining the above results, we have dropped all terms containing $\sim s^2xy^3$ and $\sim s^2x^3y$, as use of the identities:

$$\text{(i) } (1-\alpha^2)xy^3 = \alpha y\left(\frac{\partial W}{\partial x}\right) - x\left(\frac{\partial W}{\partial y}\right) + \mathcal{O}(s)\ldots \tag{A.6}$$

$$\text{(ii) } (1-\alpha^2)x^3y = \alpha x\left(\frac{\partial W}{\partial y}\right) - y\left(\frac{\partial W}{\partial x}\right) + \mathcal{O}(s)\ldots \tag{A.7}$$

tell us that the contributions from such terms vanish on partial integration.

The final result follows by carrying out similar reduction procedures for the other terms. A few other important identities that we shall require in our simplifications are listed below.

$$\text{(i) } \alpha x^2y^3 = xy\left(\frac{\partial W}{\partial x}\right) - x^4y - 2s\beta_1x^2y^2 - s\beta_2xy^3 + \mathcal{O}(s^2)\ldots \tag{A.8}$$

$$\text{(ii) } \alpha x^3y^2 = xy\left(\frac{\partial W}{\partial y}\right) - xy^4 - 2s\beta_2x^2y^2 - s\beta_1x^3y + \mathcal{O}(s^2)\ldots \tag{A.9}$$

$$\text{(iii) } \left(1-\alpha^2\right)x^4y = xy\left(\frac{\partial W}{\partial x}\right) - \alpha x^2\left(\frac{\partial W}{\partial y}\right) - s\beta_2xy^3 - 2s\beta_1x^2y^2 + \mathcal{O}(s^2)\ldots \tag{A.10}$$

$$\text{(iv) } \left(1-\alpha^2\right)xy^4 = xy\left(\frac{\partial W}{\partial y}\right) - \alpha y^2\left(\frac{\partial W}{\partial x}\right) - s\beta_1x^3y - 2s\beta_2x^2y^2 + \mathcal{O}(s^2)\ldots. \tag{A.11}$$

All these identities are constructed aiming at expressing the chiral fields with dimension > 1, in terms of fields of lower dimensions (by pulling out factors of s), modulo the equations of motion (namely $\partial_{x,y}W = 0$).

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Corrigendum

Quantum eigenfunctions in terms of periodic orbits of chaotic systems
Agam O and Fishman S 1993 *J. Phys. A: Math. Gen.* **26** 2113–2137

In this paper semiclassical expressions for the Wigner functions corresponding to eigenstates were obtained. Each of these consists of a contribution from classical periodic orbits as well as a smooth part. The main subject of the paper was the investigation of the contribution from the periodic orbits, and unfortunately the contribution of the smooth part was overlooked. Therefore several formulae that are presented in this paper should be corrected. Equation (3.24) should be replaced by a more accurate one, corresponding to equation (16) of Berry (1989). The various arguments leading to the main result, namely (3.64), still hold but a smooth term has to be added to this equation. The resulting correct equation is,

$$W_\alpha(x) \approx \frac{\pi A(x, E_\alpha)}{h^2\Delta'(E_\alpha)}\Delta_i(E_\alpha) + \frac{4\pi}{h^2\Delta'(E_\alpha)}\mathrm{Re}\left\{-\mathrm{i}\sum_{p,n,\mu} A^p(x, E_\alpha)c_\mu^{(p,n)} \times g_p^{(n)}(b)\exp^{-\mathrm{i}\pi\bar{N}(E_\alpha)+\frac{\mathrm{i}}{\hbar}S_\mu^p(x)}\mathrm{Erfc}\left\{\frac{\xi_p(\mu, x, E_\alpha)}{B(K,\hbar, E_\alpha)\sqrt{2\hbar}}\right\}\right\}$$

where the first term is the smooth contribution while the second was obtained in the original paper. The smooth term depends on the Airy factor $A(x, E_\alpha)$ that is just $A^p(x, E_\alpha)$ defined by (3.8) and on $\Delta_i(E)$ that is the imaginary part of the sum on the RHS of (2.15). Consequently (3.63) for the smooth part should be disregarded. The equations derived from (3.64) should be corrected as well. A contribution resulting from the smooth term should be added to (4.3). The required integral of the Airy factor over the momenta is given by equations (23),(24) of Berry (1989). A smooth term $\frac{T_p\Delta_i(E_\alpha)}{2\hbar\Delta'(E_\alpha)}$ should be added to the RHS of (5.2). The scar weight should be measured with reference to the smooth background, and the contribution of the background should be subtracted from the integral of $W_\alpha(x)$ over the tube Γ_p which is of volume $T_p h$ on the energy surface. The equality $Y_p(E_\alpha) = \frac{T_p}{\hbar\Delta'(E_\alpha)}\Lambda_p(E_\alpha)$ still holds but it is no more equal to the integral in (5.4). A contribution

$$\bar{O} = \frac{\pi\Delta_i(E_\alpha)}{h^2\Delta'(E_\alpha)}\int \mathrm{d}x\, A(x, E_\alpha)O(x)$$

should be added to the RHS of (5.10). Finally, a term $\hbar\pi\bar{d}(E_\alpha)\Delta_i(E_\alpha)$ should be added to the RHS of the sum rule (5.12).

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